

Contravariant symbol quantization on S^2

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Abstract

We define an algebra of contravariant symbols on S^2 and give an algebraic proof of the Correspondence Principle for that algebra.

§0. Introduction.

In [1] F.A.Berezin introduced a general concept of quantization on a symplectic manifold Ω . To define a quantization on Ω one needs the following data. Let F be a set of positive numbers with a limit point 0. For each $h \in F$ let $\mathcal{A}_h \subset C^\infty(\Omega)$ be an algebra with multiplication $*_h$ such that for $h < h'$, $\mathcal{A}_h \supset \mathcal{A}_{h'}$ as linear spaces. Denote $\mathcal{A} = \cup \mathcal{A}_h$. Assume that for each $h \in F$ there is given a representation of \mathcal{A}_h in some Hilbert space H_h . These data define a quantization on Ω if the Correspondence Principle holds, i.e. for $f, g \in \mathcal{A}$

$$\lim_{h \rightarrow 0} f *_h g = fg; \quad \lim_{h \rightarrow 0} h^{-1}(f *_h g - g *_h f) = i\{f, g\},$$

where $\{\cdot, \cdot\}$ denotes a Poisson bracket on Ω . If for $f \in \mathcal{A}_h$ \hat{f} denotes a corresponding operator in H_h , the function f is called a symbol of \hat{f} .

Thus to define a quantization one may start from an appropriate construction of symbols. In [2] Berezin introduced covariant and contravariant operator symbols and extensively investigated their various properties. In [3] he applied covariant symbols to quantization of Kähler manifolds. A particular example of covariant symbol quantization on S^2 was considered in [1].

Therein Berezin described the algebra of covariant symbols and gave an analytic proof of the Correspondence Principle for covariant symbol quantization on S^2 .

A more advanced approach to quantization as to deformation of classical mechanics was developed in [4].

In this paper we will define algebras of co- and contravariant symbols on S^2 , two of them in the same framework and give an algebraic proof of the Correspondence Principle both for co- and contravariant symbol quantizations.

§1. Covariant and contravariant symbols on S^2 .

Consider a Hilbert space $L^2(\mathbf{C}, d\alpha_n)$, with a measure

$$d\alpha_n(z, \bar{z}) = \frac{n}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^{n+1}}$$

and with the scalar product denoted by (\cdot, \cdot) . Let H_n be the n -dimensional subspace of $L^2(\mathbf{C}, d\alpha_n)$ of all polynomials in z of degree $\leq n-1$. For $v \in \mathbf{C}$ the vectors $e_{\bar{v}}(z) = (1 + z\bar{v})^{n-1} \in H_n$ have a following reproducing property, for $f \in H_n$ $f(v) = (f, e_{\bar{v}})$.

Definition. The covariant symbol of an operator $A \in H_n$ is the function $f(z, \bar{z}) = (Ae_{\bar{z}}, e_{\bar{z}})/(e_{\bar{z}}, e_{\bar{z}})$.

To define a contravariant symbol one needs a notion of the canonical measure μ_n on \mathbf{C} ,

$$d\mu_n(z, \bar{z}) = (e_{\bar{z}}, e_{\bar{z}})d\alpha_n(z, \bar{z}) = \frac{n}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}.$$

Let $P_{z, \bar{z}}$ denote the orthogonal projection operator on $e_{\bar{z}}$ in H_n .

Definition. A function $f(z, \bar{z})$ is a contravariant symbol of the operator A if

$$A = \int f(z, \bar{z}) P_{z, \bar{z}} d\mu_n(z, \bar{z}).$$

Let $G = SU(2)$ be the group of all unitary 2×2 -matrices with the determinant 1. Let G act on \mathbf{C} from the right by fractional-linear transformations, for

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in G, \tag{1}$$

$z \in \mathbf{C} \quad g : z \mapsto zg = (az - \bar{b})/(bz + \bar{a})$. Actually G acts on the widened complex plane, i.e. on the Riemann sphere S^2 by rigid rotations. The canonical measure μ_n , considered as a measure on S^2 , is rotation-invariant.

For each natural n the group G has exactly one unitary n -dimensional representation up to unitary equivalence. Denote it by π_n . There exists a realization of π_n in H_n as follows. For g given by (1) and $f \in H_n$, one has $(\pi_n(g)f)(z) = (bz + \bar{a})^{n-1}f(zg)$. Since $\pi_n(g)e_{\bar{v}} = (\bar{a} - \bar{b}\bar{v})^{n-1}e_{\bar{v}g^{-1}}$, one immediately finds that both for covariant and contravariant symbols, the symbol — operator correspondence is G -equivariant, i.e. if $f(z, \bar{z})$ is a symbol of an operator A in H_n then $f(zg, \bar{z}g)$ is a symbol of $\pi_n(g)A\pi_n(g^{-1})$. Thus it is natural to consider both covariant and contravariant symbols as functions on S^2 . In particular to define a covariant symbol at infinity one needs to replace $e_{\bar{z}}$ by $e_{\infty} = z^{n-1}$ in the definition of a covariant symbol. A nice invariant way to introduce the so called coherent states $\{e_{\bar{v}}\}$ and covariant symbols in terms of line bundles can be found in [5].

§2. Symbols corresponding to the universal enveloping algebra elements.

The Lie algebra $su(2)$ of G consists of all skew Hermitian traceless 2×2 -matrices. Its complexification is the Lie algebra $sl(2, \mathbf{C})$ of traceless complex 2×2 -matrices. Let U denote the universal enveloping algebra of $sl(2, \mathbf{C})$. The representation π_n can be defined on $su(2)$ by derivation, then extended to $sl(2, \mathbf{C})$ by complex-linearity and finally extended to U . For

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl(2, \mathbf{C})$$

one has

$$\pi_n(X) = (-bz^2 + 2az + c)\frac{d}{dz} + (n-1)(bz - a). \quad (2)$$

Thus for $u \in U$ $u_n = \pi_n(u)$ is a differential operator with polynomial coefficients in z and n . Let $s_n(u)$ denote the covariant symbol of $u \in U$. It can be calculated as follows

$$s_n(u) = \frac{(u_n e_{\bar{z}}, e_{\bar{z}})}{(e_{\bar{z}}, e_{\bar{z}})} = \frac{(u_n e_{\bar{z}})(z)}{e_{\bar{z}}(z)} = \frac{u_n(1 + z\bar{z})^{n-1}}{(1 + z\bar{z})^{n-1}}. \quad (3)$$

Observe that the symbol $s_n(u)$ is polynomial in n .

The adjoint action Ad of G on $su(2)$ (by rotations) can be naturally extended to U . Then for $u \in U$, $g \in G$ one has $\pi_n(Ad(g)u) = \pi_n(g)\pi_n(u)\pi_n(g^{-1})$. Therefore, from G -equivariance of covariant symbols it follows that the mapping $u \mapsto s_n(u)$ is also G -equivariant, i.e., for $g \in G$ $s_n(Ad(g)u)(z, \bar{z}) = s_n(u)(zg, \overline{z\bar{g}})$.

Now we will give an explicit description of the mapping s_n using a G -module structure of U under adjoint action.

Consider elements of $sl(2, \mathbf{C})$ as complex linear functionals on $su(2)$ with respect to Ad -invariant pairing $X, Y \mapsto -\frac{1}{2}trXY$ for $X \in sl(2, \mathbf{C})$ and $Y \in su(2)$. The symmetrization mapping Sym (see [6]) is a \mathbf{C} -linear isomorphism of the algebra Λ of all complex polynomials on $su(2)$ onto U such that if $f(Y) = -\frac{1}{2}trXY$ is a functional on $su(2)$ for an arbitrary $X \in sl(2, \mathbf{C})$ then $Sym(f^k) = X^k$ for all natural k . It is G -equivariant, i.e., Sym maps $f(Ad(g^{-1})Y)$ to $Ad(g)Sym(f)$ for $f \in \Lambda$. Let I, M denote the spaces of all rotation-invariant and harmonic polynomials in Λ respectively. Then $Z = Sym(I)$ is the center of U . Denote $E = Sym(M)$. It is known that $\Lambda = I \otimes M$ and $U = Z \otimes E$ (in the both tensor products $x \otimes y$ corresponds to the respective product xy , see [7]). Thus each element $u \in U$ can be written as $u = z_1v_1 + \dots + z_kv_k$ for some $z_i \in Z, v_i \in E$.

Since π_n is irreducible for each $z \in Z$ the operator $\pi_n(z)$ is scalar. Denote that scalar by $\chi_n(z)$. The function χ_n is a homomorphism of Z into \mathbf{C} and is called a central character of U corresponding to π_n .

Lemma 1. *For $z \in Z, u \in U$*

- (i) *the symbol $s_n(z)$ is a constant equal to $\chi_n(z)$;*
- (ii) *$s_n(zu) = s_n(z)s_n(u)$.*

Proof. Since the covariant symbol of the identity operator is identically 1, the symbol of $\pi_n(z)$ is identically equal to $\chi_n(z)$, which proves (i). Now, (ii) is obvious.

In order to describe s_n on U it suffices to know the restrictions of s_n to Z and E .

The adjoint action of G on $su(2)$ keeps invariant a square of Euclidean radius, $(r(Y))^2 = -\frac{1}{2}trY^2$, $Y \in su(2)$, which is a quadratic polynomial on $su(2)$. It is known (see, e.g.[6]) that Z is a polynomial algebra in the Casimir element $z_0 = Sym(r^2)$. A direct calculation provides

Lemma 2. $s_n(z_0) = 1 - n^2$.

Let M_k denote the subspace of M of harmonic polynomials of degree k . It is known that with respect to the action of G on Λ , via a change of variables, M_k is a $(2k+1)$ -dimensional irreducible subspace. Let v_0 denote the element of U corresponding to

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in sl(2, \mathbf{C}),$$

$f_0(Y) = -\frac{1}{2}tr VY$. Then for each natural k $Sym(f_0^k) = v_0^k$. It is easy to check directly that f_0^k is harmonic, so $f_0^k \in M_k$. Using (2) one gets that $(v_0^k)_n = (\frac{d}{dz})^k$ for all n . Now from (3) follows

$$\text{Lemma 3. } s_n(v_0^k) = (n-1)(n-2)\dots(n-k)\left(\frac{\bar{z}}{1+z\bar{z}}\right)^k.$$

Consider a G -equivariant embedding of S^2 in $su(2)$ given as follows

$$S^2 \supset \mathbf{C} \ni z \mapsto \begin{pmatrix} i\frac{1-z\bar{z}}{1+z\bar{z}} & -2i\frac{\bar{z}}{1+z\bar{z}} \\ -2i\frac{z}{1+z\bar{z}} & -i\frac{1-z\bar{z}}{1+z\bar{z}} \end{pmatrix} \in su(2).$$

The image of S^2 is the unit sphere with respect to the Euclidean scalar product $X, Y \mapsto -\frac{1}{2}tr XY$ in $su(2)$. Then the pullback of $f_0(Y)$ to S^2 is $i\frac{\bar{z}}{1+z\bar{z}}$. Thus identifying S^2 with the unit sphere in $su(2)$ one gets

$$s_n(Sym(f_0^k)) = s_n(v_0^k) = \frac{1}{i^k}(n-1)(n-2)\dots(n-k)f_0^k|_{S^2}. \quad (4)$$

Since all the ingredients of (4) are G -equivariant, one can replace f_0^k in (4) by a linear combination of its rotations by the elements of G . Since G acts irreducibly in M_k one thus obtains an arbitrary element of M_k .

$$\text{Lemma 4. For all } f \in M_k \text{ } s_n(Sym(f)) = \frac{1}{i^k}(n-1)(n-2)\dots(n-k)f|_{S^2}.$$

Denote $E_k = Sym(M_k)$. Since $M = \oplus_k M_k$ then $E = \oplus_k E_k$. Therefore, an arbitrary element of U may be expressed as a sum of monomials of a form $z_0^j v$ with $v \in E_k$. Combining Lemmas 1 - 4, one gets

Proposition 1. *Let $v \in E_k$, $v = Sym(f)$ for some $f \in M_k$. Then*

$$s_n(z_0^j v) = \left(\frac{1}{i}\right)^{2j+k}(n^2-1)^j(n-1)(n-2)\dots(n-k)f|_{S^2}.$$

§3. Symbol algebras.

Let R denote the space of restrictions of all polynomials from Λ to the unit sphere S^2 . The elements of R are called regular functions on S^2 . It is known (see, e.g.[8]) that the restriction of the space M of harmonic polynomials to S^2 is a bijection of M onto R . Therefore each regular function on S^2 is a restriction of a unique harmonic polynomial. Denote by R_k the restriction of M_k . Thus $R = \oplus_k R_k$.

Since for all $u \in U$ $s_n(u)$ is polynomial in n one can consider $s_t(u)$ for arbitrary $t \in \mathbf{C}$. It is obvious that Lemma 2 is valid for $s_t(u)$ for all complex t . Namely the mapping $z \mapsto s_t(z)$ is a homomorphism of Z to \mathbf{C} and for $z \in Z, u \in U$ $s_t(zu) = s_t(z)s_t(u)$.

Denote $\mathcal{A}_{1/t} = s_t(U)$. In the sequel \mathbf{N}^* will denote the set of all positive integers. From Proposition 1 immediately follows

Proposition 2. *For $t = n \in \mathbf{N}^*$ $\mathcal{A}_{1/t} = \oplus_{k=0}^{k=n-1} R_k$. For all other values of t $\mathcal{A}_{1/t}$ consists of all regular functions.*

We are going to show that the kernel of the mapping s_t is a two-sided ideal in U , thus obtaining a quotient algebra structure in $\mathcal{A}_{1/t}$.

Let J_t be the two-sided ideal in U generated by $Z \cap \text{Ker } s_t$.

Lemma 5. $U = E + J_t$.

Proof. For $u = zv$ with $z \in Z, v \in E$ one has $u = s_t(z)v + (z - s_t(z))v$ where $s_t(z)$ is identified with the respective constant in U . The assertion of Lemma follows from the fact that $z - s_t(z) \in Z \cap \text{Ker } s_t$.

Lemma 6. *For $t \notin \mathbf{N}^*$ $\text{Ker } s_t = J_t$.*

Proof. From Lemma 5 follows that $\text{Ker } s_t = E \cap \text{Ker } s_t + J_t$. Since the restriction of the space of harmonic polynomials to a sphere is a bijection onto the space of regular functions, it follows from Lemma 4 that $E \cap \text{Ker } s_t$ is trivial for $t \notin \mathbf{N}^*$.

Proposition 3. *For all $t \in \mathbf{C}$ $\text{Ker } s_t$ is a two-sided ideal in U .*

Proof. For $t = n \in \mathbf{N}^*$ $\text{Ker } s_t = \text{Ker } \pi_n \subset U$. For the rest of t Lemma 6 is applied.

Now $\mathcal{A}_{1/t}$ carries a quotient algebra structure. Denote the corresponding multiplication in $\mathcal{A}_{1/t}$ by $*_{1/t}$.

A following Lemma is obtained from direct calculations.

Lemma 7. *The function $f(z, \bar{z}) = (n+1)(n+2)\dots(n+k)(\frac{\bar{z}}{1+z\bar{z}})^k$ is a contravariant symbol of the operator $\pi_n(v_0^k) = (\frac{d}{dz})^k$ in H_n .*

Proposition 4. *For $n \in \mathbf{N}^*$, $u \in U$ the function $s_{-n}(u)(-1/\bar{z}, -1/z)$ is a contravariant symbol of the operator $\pi_n(u)$ in H_n .*

Proof. It follows from Lemma 2 that s_n coincides with s_{-n} on the center Z of U . Therefore the ideals J_n and J_{-n} coincide as well. Since $J_n \subset \text{Ker } \pi_n$ for each $u \in J_{-n}$ both the symbol $s_{-n}(u)$ and operator $\pi_n(u)$ are zero. It follows from Lemma 5, that it remains to check the assertion of the Proposition for $u \in E_k$, since $E = \oplus E_k$. The rest follows from Lemma 7, the irreducibility of E_k with respect to the adjoint action of G and equivariance of contravariant symbols.

Thus the algebra $\mathcal{A}_{-1/n}$ consists of contravariant symbols of all operators in H_n up to the antipodal mapping $z \mapsto -1/\bar{z}$ of the sphere S^2 . Moreover, the mapping which maps the symbol $s_{-n}(u)$ to the operator $\pi_n(u)$ in H_n is a correctly defined homomorphism of $\mathcal{A}_{-1/n}$ onto $\text{End } H_n$.

§4. The proof of the Correspondence Principle.

Recall now some basic facts about filtration in the universal enveloping algebra and Poisson structure in the symmetric algebra of a Lie algebra (see, e.g. [6]).

Let U_k denote the subspace of U spanned by monomials of degree $\leq k$. Then $\{U_k\}$ is a filtration, for $u \in U_k, v \in U_l$ both $uv, vu \in U_{k+l}$ and $uv - vu \in U_{k+l-1}$.

Let Λ_k denote the subspace of Λ of homogenous polynomials of degree k . Then $\text{Sym}(\Lambda_k) \subset U_k$. Moreover, Sym composed with the quotient mapping $U_k \rightarrow U_k/U_{k-1}$ establishes an isomorphism of Λ_k onto U_k/U_{k-1} . For $u \in U_k$ let \underline{u} denote the unique element of Λ_k such that $\text{Sym}(\underline{u}) \equiv u \pmod{U_{k-1}}$.

There exists a natural Poisson structure on Λ such that if $f_i(Y) = -\frac{1}{2}\text{tr} X_i Y$, $i = 1, 2, 3$ are linear functionals on $\mathfrak{su}(2)$ corresponding to $X_i \in \mathfrak{sl}(2, \mathbf{C})$ with $[X_1, X_2] = X_3$, then $\{f_1, f_2\} = f_3$. The symplectic leaves of that Poisson structure are the G -orbits in $\mathfrak{su}(2)$, i.e. the spheres. Denote by

$\{\cdot, \cdot\}_{S^2}$ the restriction of the Poisson bracket to the unit sphere S^2 . Then for $f, g \in \Lambda$ $\{f|_{S^2}, g|_{S^2}\}_{S^2} = \{f, g\}|_{S^2}$.

For $u \in U_k, v \in U_l$ $\underline{uv} = \underline{vu} = \underline{u} \cdot \underline{v}$ while $\underline{uv} - \underline{vu} = \{\underline{u}, \underline{v}\}$.

Proposition 5. *Let $u \in U_k$. Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t^k} s_t(u) = \frac{1}{i^k} \underline{u}|_{S^2}.$$

Proof. If $f \in \Lambda_k$ and $u = \text{Sym}(f) \in U_k$, then $\underline{u} = f$. In particular $z_0 \in U_2$, $\underline{z_0} = r^2$ and the restriction of r^2 to the unit sphere S^2 is identically 1. Now the proof follows from Proposition 1.

Let f, g be regular functions on S^2 . From Proposition 2 follows that for $t \notin \mathbf{N}^*$ or sufficiently big $t = n \in \mathbf{N}^*$ the product $f *_{1/t} g$ is defined.

Theorem. *For any regular functions f, g on S^2 holds*

$$\lim_{t \rightarrow \infty} f *_{1/t} g = fg; \quad \lim_{t \rightarrow \infty} t(f *_{1/t} g - g *_{1/t} f) = i\{f, g\}_{S^2}.$$

Proof. It is enough to consider $f \in R_k, g \in R_l$. Let $u \in E_k, v \in E_l$ be such that $\frac{1}{i^k} \underline{u}$ and $\frac{1}{i^l} \underline{v}$ are the harmonic extensions of f and g , respectively. Then, using Proposition 5 one gets

$$f \cdot g = \frac{1}{i^k} \underline{u}|_{S^2} \cdot \frac{1}{i^l} \underline{v}|_{S^2} = \frac{1}{i^{k+l}} \underline{uv}|_{S^2} = \lim_{t \rightarrow \infty} \frac{1}{t^{k+l}} s_t(uv) = \lim_{t \rightarrow \infty} \frac{1}{t^{k+l}} s_t(u) *_{1/t} s_t(v).$$

Applying Lemma 4 to the last expression one finally obtains

$$f \cdot g = \lim_{t \rightarrow \infty} \frac{(t-1) \dots (t-k)(t-1) \dots (t-l)}{t^{k+l}} f *_{1/t} g = \lim_{t \rightarrow \infty} f *_{1/t} g.$$

Proceeding in a similar manner one gets

$$\begin{aligned} i\{f, g\}_{S^2} &= i\left\{\frac{1}{i^k} \underline{u}|_{S^2}, \frac{1}{i^l} \underline{v}|_{S^2}\right\}_{S^2} = \frac{1}{i^{k+l-1}} \{\underline{u}, \underline{v}\}|_{S^2} = \frac{1}{i^{k+l-1}} \underline{uv - vu}|_{S^2} = \\ &= \lim_{t \rightarrow \infty} \frac{1}{t^{k+l-1}} s_t(uv - vu) = \lim_{t \rightarrow \infty} \frac{(t-1) \dots (t-k)(t-1) \dots (t-l)}{t^{k+l}} t(f *_{1/t} g - g *_{1/t} f) = \\ &= \lim_{t \rightarrow \infty} t(f *_{1/t} g - g *_{1/t} f). \end{aligned}$$

Let $F = \{1, 1/2, 1/3, \dots\}$. According to the Theorem, for $h = 1/n \in F$ the algebras $\mathcal{A}_{1/n}$ and $\mathcal{A}_{-1/n}$ of covariant and contravariant symbols of operators in H_n form the data for covariant and contravariant quantization on S^2 respectively.

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